Circle Coordinate Geometry 2

1.

a The radius of the circle is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(4-1)^2+(6-(-2))^2}=\sqrt{3^2+8^2}=\sqrt{9+64}=\sqrt{73}$$

The equation of the circle is

$$(x-4)^2+(y-6)^2=\left(\sqrt{73}\right)^2$$

or $(x-4)^2+(y-6)^2=73$

b The gradient of the line joining (1, –2) and (4, 6) is

$$\frac{y_2-y_1}{x_2-x_1}=\frac{6-(-2)}{4-1}=\frac{6+2}{3}=\frac{8}{3}$$

The gradient of the tangent is $\frac{-1}{\left(\frac{8}{3}\right)}=-\frac{3}{8}$

The equation of the tangent to the circle at (1, –2) is

$$y-y_1=m(x-x_1)$$

$$y-(-2)=-\frac{3}{8}(x-1)$$

$$y+2=-\frac{3}{8}(x-1)$$

$$8y+16=-3(x-1)$$

$$8y+16=-3x+3$$

$$3x+8y+16=3$$

$$3x+8y+13=0$$

2.

a Substitute $x = -7$ and $y = -6$ into $x^2 + 18x + y^2 - 2y + 29$

$$\begin{aligned} x^2 + 18x + y^2 - 2y + 29 &= (-7)^2 + 18(-7) + (-6)^2 - 2(-6) + 29 \\ &= 49 - 126 + 36 + 12 + 29 \\ &= 0 \end{aligned}$$

The point P satisfies the equation, so P lies on C.

b Completing the square gives

$(x + 9)^2 + (y - 1)^2 = 53$

The centre of the circle, A, is $(-9, 1)$.

The gradient of $CP = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-6-1}{-7-(-9)} = \dfrac{-7}{2}$

Gradient of the tangent is $\dfrac{2}{7}$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-6) &= \frac{2}{7}(x - (-7)) \\ y &= \frac{2}{7}x - 4 \end{aligned}$$

c The tangent intersects the y-axis at $x = 0$

$y = \dfrac{2}{7}(0) - 4 = -4$

$R(0, -4)$

d Height of triangle = radius of circle = $\sqrt{53}$

Base of triangle = distance PR

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-7-0)^2 + (-6-(-4))^2} = \sqrt{53}$

Area $APR = \dfrac{1}{2} \times \sqrt{53} \times \sqrt{53} = 26.5$ units^2

3.

a The centre of the circle $(x+4)^2+(y-1)^2=242$ is $(-4,1)$.
The gradient of the line joining (–4, 1) and (7, –10) is
$$\frac{y_2-y_1}{x_2-x_1}=\frac{-10-1}{7-(-4)}=\frac{-11}{7+4}=-\frac{11}{11}=-1$$

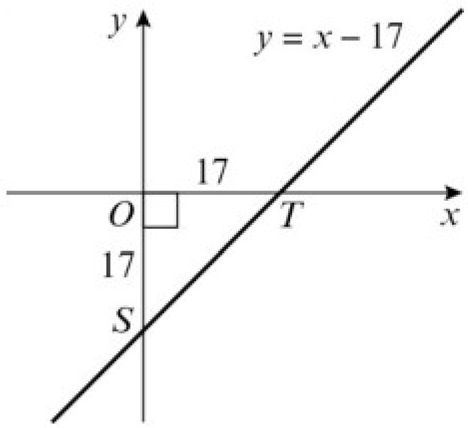
The gradient of the tangent is $\frac{-1}{(-1)}=1$.

The equation of the tangent is
$$y-y_1=m(x-x_1)$$
$$y-(-10)=1(x-7)$$
$$y+10=x-7$$
$$y=x-17$$

Substitute $x = 0$ into $y = x - 17$
$y = 0 - 17$
$y = -17$
So the coordinates of S are (0, –17)

Substitute $y = 0$ into $y = x - 17$
$0 = x - 17$
$x = 17$
So the coordinates of T are (17, 0).

b



The area of ΔOST is $\frac{1}{2}\times17\times17=144.5$

4.

a The centre of the circle, Q, is $(11, -5)$

To find the radius of the circle:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(5-11)^2 + (3-(-5))^2}$$
$$= \sqrt{(-6)^2 + 8^2}$$
$$= 10$$

$(x - 11)^2 + (y + 5)^2 = 100$

b The gradient of $PQ = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-(-5)}{5-11} = \dfrac{4}{-3}$

Gradient of the tangent is $\dfrac{3}{4}$

$$y - y_1 = m(x - x_1)$$
$$y - 3 = \frac{3}{4}(x - 5)$$
$$y = \frac{3}{4}x - \frac{3}{4}$$

c Midpoint of $PQ = \left(\dfrac{11+5}{2}, \dfrac{-5+3}{2}\right) = (8, -1)$

Gradient of l_2 is $\dfrac{3}{4}$ as the line is parallel to l_1.

$$y - y_1 = m(x - x_1)$$
$$y - (-1) = \frac{3}{4}(x - 8)$$
$$y = \frac{3}{4}x - 7$$

c Solve $y = \dfrac{3}{4}x - 7$ and $(x - 11)^2 + (y + 5)^2 = 100$ simultaneously

$$(x - 11)^2 + \left(\frac{3}{4}x - 2\right)^2 = 100$$
$$x^2 - 22x + 121 + \frac{9}{16}x^2 - 3x + 4 - 100 = 0$$
$$25x^2 - 400x + 400 = 0$$
$$x^2 - 16x + 16 = 0$$

Using the formula, $x = \dfrac{16 \pm \sqrt{192}}{2} = 8 \pm 4\sqrt{3}$

When $x = 8 + 4\sqrt{3}$, $y = \dfrac{3}{4}(8 + 4\sqrt{3}) - 7 = -1 + 3\sqrt{3}$

When $x = 8 - 4\sqrt{3}$, $y = \dfrac{3}{4}(8 - 4\sqrt{3}) - 7 = -1 - 3\sqrt{3}$

$A(8 - 4\sqrt{3}, -1 - 3\sqrt{3})$ and $B(8 + 4\sqrt{3}, -1 + 3\sqrt{3})$

d $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(8 + 4\sqrt{3} - (8 - 4\sqrt{3}))^2 + (-1 + 3\sqrt{3} - (-1 - 3\sqrt{3}))^2}$

$= \sqrt{(8\sqrt{3})^2 + (6\sqrt{3})^2}$

$= \sqrt{192 + 108}$

$= \sqrt{300}$

$= 10\sqrt{3}$

5.

a $A(-1, -9)$ and $B(7, -5)$

Midpoint $= \left(\frac{-1+7}{2}, \frac{-9+(-5)}{2}\right) = (3, -7)$

The gradient of the line segment $AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-(-9)}{7-(-1)} = \frac{1}{2}$

So the gradient of a line perpendicular to AB is -2.
The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = -2$ and $(x_1, y_1) = (3, -7)$

So $y - (-7) = -2(x - 3)$

$y = -2x - 1$

b Centre of the circle is $(1, -3)$
Substitute $x = 1$ into the equation $y = -2x - 1$
$y = -2(1) - 1 = -3$
Therefore, the perpendicular bisector of AB, $y = -2x - 1$, passes through the centre of the circle $(1, -3)$

6.

$(x + 5)^2 + (y + 3)^2 = 80$
Gradient of tangent = 2, so $y = 2x + c$

Diameter of the circle that touches l_1 and l_2 has gradient $-\frac{1}{2}$ and passes through the centre of the circle $(-5, -3)$

$y = -\frac{1}{2}x + d$

$-3 = -\frac{1}{2}(-5) + d$

$d = -\frac{11}{2}$

$y = -\frac{1}{2}x - \frac{11}{2}$ is the equation of the diameter that touches l_1 and l_2.

Solve the equation of the diameter and circle simultaneously:

$$(x + 5)^2 + (-\frac{1}{2}x - \frac{5}{2})^2 = 80$$

$$x^2 + 10x + 25 + \frac{1}{4}x^2 + \frac{5}{2}x + \frac{25}{4} - 80 = 0$$

$$4x^2 + 40x + 100 + x^2 + 10x + 25 - 320 = 0$$

$$5x^2 + 50x - 195 = 0$$

$$x^2 + 10x - 39 = 0$$

$$(x + 13)(x - 3) = 0$$

$x = -13$ or $x = 3$

When $x = -13$, $y = -\frac{1}{2}(-13) - \frac{11}{2} = 1$

When $x = 3$, $y = -\frac{1}{2}(3) - \frac{11}{2} = -7$

$(-13, 1)$ and $(3, -7)$ are the coordinates where the diameter touches lines l_1 and l_2.

Substitute these coordinates into the equation $y = 2x + c$

When $x = -13$, $y = 1$, $1 = 2(-13) + c$, $c = 27$, $y = 2x + 27$
When $x = 3$, $y = -7$, $-7 = 2(3) + c$, $c = -13$, $y = 2x - 13$
l_1: $y = 2x + 27$
l_2: $y = 2x - 13$

7.

a $x^2 - 4x + y^2 - 6y = 7$

$(x - 2)^2 - 4 + (y - 3)^2 - 9 = 7$

$(x - 2)^2 + (y - 3)^2 = 20$

Substitute $x = 3y - 17$ into $(x - 2)^2 + (y - 3)^2 = 20$

$(3y - 19)^2 + (y - 3)^2 = 20$

$9y^2 - 114y + 361 + y^2 - 6y + 9 - 20 = 0$

$10y^2 - 120y + 350 = 0$

$y^2 - 12y + 35 = 0$

$(y - 7)(y - 5) = 0$

$y = 7$ or 5

when $y = 7$, $x = 3(7) - 17 = 4$

when $y = 5$, $x = 3(5) - 17 = -2$

$P(-2, 5)$ and $Q(4, 7)$

b Centre of circle $T = (2, 3)$ and $P(-2, 5)$

Gradient of $PT = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{5-3}{-2-2} = -\dfrac{1}{2}$

Gradient of the tangent is 2

$y - y_1 = m(x - x_1)$

$y - 5 = 2(x + 2)$

$y = 2x + 9$

Gradient of $QT = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7-3}{4-2} = 2$

Gradient of the tangent is $-\dfrac{1}{2}$

$y - y_1 = m(x - x_1)$

$y - 7 = -\dfrac{1}{2}(x - 4)$

$y = -\dfrac{1}{2}x + 9$

c Gradient $PQ = \dfrac{7-5}{4-(-2)} = \dfrac{1}{3}$

Midpoint of $PQ = \left(\dfrac{-2+4}{2}, \dfrac{5+7}{2}\right) = (1, 6)$

Gradient of the perpendicular bisector is -3

$y - y_1 = m(x - x_1)$

$y - 6 = -3(x - 1)$

$y = -3x + 9$

d $y = 2x + 9$, $y = -\frac{1}{2}x + 9$ and $y = -3x + 9$

Solve $y = 2x + 9$ and $y = -\frac{1}{2}x + 9$ simultaneously

$2x + 9 = -\frac{1}{2}x + 9$

$4x + 18 = -x + 18$

$x = 0, y = 9$

$(0, 9)$

Solve $y = 2x + 9$ and $y = -3x + 9$ simultaneously

$2x + 9 = -3x + 9$

$x = 0, y = 9$

$(0, 9)$

Therefore all three lines intersect at $(0, 9)$

8.

a $U(-2, 8)$, $V(7, 7)$ and $W(-3, -1)$

$UV^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$= (7+2)^2 + (7-8)^2$

$= 82$

$VW^2 = (-3-7)^2 + (-1-7)^2$

$= 164$

$UW^2 = (-3+2)^2 + (-1-8)^2$

$= 82$

Use Pythagoras' theorem to show $UV^2 + UW^2 = VW^2$

$82 + 82 = 164 = VW^2$

Therefore, UVW is a right-angled triangle.

b UVW is a right-angled triangle, therefore VW is the diameter of the circle.

Centre of circle = Midpoint of VW

Midpoint $= \left(\frac{7+(-3)}{2}, \frac{7+(-1)}{2}\right) = (2, 3)$

c Radius of the circle is $\frac{1}{2}$ of $VW = \frac{\sqrt{164}}{2} = \sqrt{\frac{164}{4}} = \sqrt{41}$

$(x - 2)^2 + (y - 3)^2 = 41$

9.

a **i** $A(-3, 19)$ and $B(9, 11)$

Midpoint $= \left(\frac{-3+9}{2}, \frac{19+11}{2}\right) = (3, 15)$

The gradient of the line segment $AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11-19}{9-(-3)} = -\frac{2}{3}$

So the gradient of the line perpendicular to AB is $\frac{3}{2}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = \frac{3}{2}$ and $(x_1, y_1) = (3, 15)$

So $y - 15 = \frac{3}{2}(x - 3)$

$y = \frac{3}{2}x + \frac{21}{2}$

ii $A(-3, 19)$ and $C(-15, 1)$

Midpoint $= \left(\frac{-3-15}{2}, \frac{19+1}{2}\right) = (-9, 10)$

The gradient of the line segment $AC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-19}{-15+3} = \frac{3}{2}$

So the gradient of the line perpendicular to AC is $-\frac{2}{3}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = -\frac{2}{3}$ and $(x_1, y_1) = (-9, 10)$

So $y - 10 = -\frac{2}{3}(x + 9)$

$y = -\frac{2}{3}x + 4$

b Solve $y = \frac{3}{2}x + \frac{21}{2}$ and $y = -\frac{2}{3}x + 4$ simultaneously

$\frac{3}{2}x + \frac{21}{2} = -\frac{2}{3}x + 4$

$9x + 63 = -4x + 24$

$13x = -39$

$x = -3, y = -\frac{2}{3}(-3) + 4 = 6$

So, the coordinates of the centre of the circle are $(-3, 6)$

c Radius = distance from $(-3, 6)$ to $(9, 11)$

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(9+3)^2 + (11-6)^2} = \sqrt{12^2 + 5^2} = 13$

$(x + 3)^2 + (y - 6)^2 = 169$

10.

a $A(-1, 9)$, $B(6, 10)$, $C(7, 3)$, $D(0, 2)$

The length of AB is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(6-(-1))^2+(10-9)^2}=\sqrt{7^2+1^2}=\sqrt{49+1}=\sqrt{50}$$

The length of BC is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(7-6)^2+(3-10)^2}=\sqrt{1^2+(-7)^2}=\sqrt{1+49}=\sqrt{50}$$

The length of CD is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(0-7)^2+(2-3)^2}=\sqrt{(-7)^2+(-1)^2}=\sqrt{49+1}=\sqrt{50}$$

The length of DA is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(-1-0)^2+(9-2)^2}=\sqrt{(-1)^2+7^2}=\sqrt{1+49}=\sqrt{50}$$

The sides of the quadrilateral are equal.

The gradient of AB is

$$\frac{y_2-y_1}{x_2-x_1}=\frac{10-9}{6-(-1)}=\frac{1}{7}$$

a The gradient of BC is

$$\frac{y_2-y_1}{x_2-x_1}=\frac{3-10}{7-6}=\frac{-7}{1}=-7$$

The product of the gradients $=\left(\frac{1}{7}\times-7\right)=-1.$

So the line AB is perpendicular to BC.
So the quadrilateral $ABCD$ is a square.

b The area $=\sqrt{50}\times\sqrt{50}=50$

c The mid-point of AC is

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{-1+7}{2},\frac{9+3}{2}\right)=\left(\frac{6}{2},\frac{12}{2}\right)=(3,6)$$

So the centre of the circle is $(3, 6)$.